PARTIAL FRACTIONS

Introduction

Writing any given proper rational expression of one variable as a sum (or difference) of rational expressions whose denominators are in the simplest forms is called the partial fraction decomposition.

A proper rational expression is a rational expression whose denominator polynomial has a degree bigger than the degree of the numerator polynomial.

When a rational expression, for which we seek Partial Fraction Decomposition, is not a proper rational expression, first we need to divide the expression using long division. Next, we find the partial fraction decomposition of remainder fraction.

Some examples with numerical fractions:

You can verify that these are, in fact, partial fraction forms:

\[
\frac{19}{15} = \frac{19}{3 \times 5} = 1 + \frac{2}{3} - \frac{2}{5}
\]

\[
\frac{7}{30} = \frac{7}{2 \times 3 \times 5} = \frac{1}{2} + \frac{2}{3} - \frac{3}{5}
\]

\[
\frac{5}{18} = \frac{5}{2 \times 3^2} = \frac{1}{2} + \frac{1}{3} - \frac{5}{3^2}
\]

Readers should note that irreducible factor condition is not relevant for numerical partial fractions. Also, partial fraction forms may not be unique for numerical examples. Moreover, the role of the linear factors in algebraic case is played by the prime number factors in the numerical case.

The simplest forms of denominator polynomials admissible are:

- linear polynomials,
- repeated powers of a linear polynomial in the case that a power of a linear factor is present in the denominator function,
- irreducible quadratic polynomials,
- repeated powers of an irreducible quadratic polynomials in the case that a power of an irreducible quadratic factor is present in the denominator function.

For instance, we seek to write the rational function

$$\frac{3x+2}{(5x+7)(9x+8)(3x+4)^3(x^2+x+2)}$$

in the form of

$$\frac{3x+2}{(5x+7)(9x+8)(3x+4)^3(x^2+x+2)} \equiv \frac{A}{5x+7} + \frac{B}{9x+8} + \frac{C}{3x+4} + \frac{D}{(3x+4)^2} + \frac{E}{(3x+4)^3} + \frac{Fx+G}{x^2+x+2}$$

What does this \( '=' \) mean? In the numerical case, the equality, \( \frac{5}{6} = \frac{1}{2} + \frac{1}{3} \), is just an equality.

However, in the algebraic case, the equality should hold for each and every feasible \( x \).
This is one reason, why partial fraction decomposition in algebraic case can be made to be unique.

The symbol, \( = \), is used to denote that the expression is an identity. That is, the equation must be true for all feasible \( x \).

Application of Partial Fractions

Partial Fraction Decomposition is useful in integration and graphing. If the rational function is not proper, first we divide the given rational function and then we apply the process of partial fraction decomposition on the newly obtained proper rational function component. In this work, I have provided a few short methods which shall be used only after the mastery over the standard procedures is achieved.

Partial Fraction Decomposition can be considered as the reverse operation of addition and subtractions of rational functions. Given a rational function, the reverse operation of writing it as a sum of other rational functions can give multitude of answers. Here, we have imposed a form that gives unique answers, which also makes anti-differentiation more accessible.
Comparison of Coefficients of an Identity

Consider the identity:

\[ Ax^3 + Bx^2 + Cx + D \equiv Px^2 + Qx + R \quad \text{(ID)} \]

Since this is an identity, the equation must hold for all values of \( x \) in the domain of this identity. This domain is the set of all real numbers. Since the identity holds for all real values of \( x \), in particular it should hold for \( x = 0 \). Therefore, substituting, \( x = 0 \), we get

\[ D = R. \]

This leads to a new identity:

\[ Ax^3 + Bx^2 + Cx + D \equiv Px^2 + Qx + R \Rightarrow Ax^3 + Bx^2 + Cx \equiv Px^2 + Qx \]

Now, consider this new identity.

\[ Ax^3 + Bx^2 + Cx \equiv Px^2 + Qx \quad \text{(*)} \]

In particular, this identity must hold for \( x = 1 \) and \( x = -1 \). (There is no magic in this choice. We only look for convenient values.) These substitutions lead to the equations,

\[ A + B + C = P + Q \]
\[ -A + B - C = P - Q, \]

Respectively. Addition of these two equations gives \( 2B = 2P \Rightarrow B = P \). Next, we substitute \( B = P \) in (*), to get

\[ Ax^3 + Cx \equiv +Qx \Rightarrow Ax^3 \equiv (Q - C)x \quad \text{(**)} \]

By substituting \( x = 1 \) and \( x = 2 \) (again there is no magic in this choice), we get

\[ A = Q - C \]
\[ 8A = 2(Q - C) \Rightarrow 8A = 2A \Rightarrow A = 0. \]

This is \( A \) by the equation above.

Finally, from this result and (**), we get, for all \( x \),

\[ (Q - C)x \equiv 0. \]

This can happen only when \( Q = C \). To see this, put \( x = 1 \).

There is another way to achieve this result, but it involves limits.
The identity (*) is true for all values of \( x \). In particular, the identity is true for nonzero values of \( x \). Therefore, we can divide (*) by \( x \), to obtain

\[
Ax^2 + Bx^2 + C \equiv Px + Q
\]

possibly except for \( x=0 \).

However, this identity must be true for all \( x \) other than zero. Therefore, this identity must be true for \( x = 0.1, \ x = 0.001, \ x = 0.0000001, \ x = 0.000000000000000001, \) etc. Again there is no magic in these values. I just needed smaller and smaller values for \( x \), to convince you that we can take the limit of both sides as \( x \) tends to zero. By applying this principle, we get

\[
\lim_{x \to 0}(Ax^2 + Bx^2 + C) = \lim_{x \to 0}(Px + Q) \Rightarrow C = Q.
\]

To get all the other coefficients, we can apply this process repeatedly.

Consider the identity given in (ID).

\[
Ax^3 + Bx^2 + Cx + D \equiv 0 \times x^3 +Px^2 + Qx + R \iff A=0, B=P, C=Q \text{ and } D=R.
\]

Even though, we have not presented a general proof of the principle of comparison of coefficients, it can be applied to any two polynomial identities. We will do so in the following examples.

**Standard Procedures (Simple Examples)**

**Example 1:** Express \( \frac{x+3}{(x+1)(x-1)} \) in partial fraction form.

This means that we need to find two constants, say \( A \) and \( B \), such that

\[
\frac{x+3}{(x+1)(x-1)} \equiv \frac{A}{x+1} + \frac{B}{x-1}.
\]

The symbol, \( \equiv \), indicates that the expression above is an identity. That is, the statement is true for all feasible values of \( x \).
\[
\frac{x+3}{(x+1)(x-1)} \equiv \frac{A}{x+1} + \frac{B}{x-1} \quad \Rightarrow \quad \frac{x+3}{(x+1)(x-1)} \equiv \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}
\]

This identity is true for all values of \(x\), except for \(x = 1\) and \(x = -1\). Why is this so? This is because of a principle you have learned in Year 3; division by zero is….

We multiply the whole equation by the least common multiple of the denominators, to get rid of the denominator. This will lead to the identity:

\[x+3 \equiv A(x-1) + B(x+1).\]

Since this is an identity, the statement is true for all values of \(x\) including \(x = -1\) and \(x = 1\). (Why is this so? Recall the limit process which we have discussed before.)

Now,

\[x = -1 \Rightarrow -1+3 = A\times(-1-1).\]

This gives \(A = -1\).

From infinitely values we can choose for \(x\), why do we choose this particular value \(x = -1\). Also,

\[x = 1 \Rightarrow 1+3 = B\times(1+1).\]

This gives \(B = 2\).

Therefore the required partial fraction form is

\[\frac{x+3}{(x+1)(x-1)} \equiv -\frac{1}{x+1} + \frac{2}{x-1}. \quad \text{(\&)}\]

**Example 2: Example of an Improper Rational Expression**

Consider the following example,

\[\frac{x^3+3}{(x+1)(x-1)}.\]

In which the degree of the numerator is bigger or equal to (in this particular case, bigger) than the degree of the denominator. Now, we obtain the following:
There is no net change in the numerator. Also, see the separation of terms, as indicated by colours. In the next step, we apply, distributive law of division, instead of doing long division.

\[
\frac{x^3 + 3}{(x+1)(x-1)} = x - x + x + 3 \quad \frac{x^3 - x + x + 3}{(x+1)(x-1)} = x^3 - x + x + 3 \quad \frac{x^3 - x + x + 3}{x^2-1}
\]

Next, we write the required partial fraction decomposition:

\[
\frac{x^3 + 3}{(x+1)(x-1)} \equiv x + \frac{x+3}{(x+1)(x-1)} \equiv x + \frac{A}{x+1} + \frac{B}{x-1}
\]

To get the partial fraction decomposition of \( \frac{x+3}{(x+1)(x-1)} \), we refer to (&), and obtain the necessary coefficients:

\[
\frac{x^3 + 3}{(x+1)(x-1)} \equiv x + \frac{x+3}{(x+1)(x-1)} \equiv x + \frac{-1}{x+1} + \frac{2}{x-1}
\]

**Example 3**

Obtain the Partial Fraction Form of \( \frac{x+3}{(x^2+1)(x-1)} \).

\[
\frac{x+3}{(x+1)(x-1)} \equiv Cx + D \quad \frac{x+3}{x^2+1} + \frac{E}{x-1} \quad (#)
\]

From this, eliminating the denominator, we obtain:

\[
x + 3 \equiv (cx + D)(x-1) + E(x^2 + 1) \quad (~)
\]

\[
x = 1 \Rightarrow 4 = 2E \rightarrow E = 2.
\]

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Put this value for $E$ in ($\sim$) to get:

$$x+3 \equiv (Cx+D)(x-1)+2(x^2+1). \quad (\sim\sim)$$

Now compare the constant term in the Left Hand Side (LHS) and in the Right Hand Side (RHS). As we have presented before, since ($\sim\sim$) is in identity, the constant terms must be the same in both sides. This can be proved by putting $x = 0$, as we have done previously.

**Know this. Try this, but do not perform this verification in exams and tests.**

*Just use the principle that, in an identity, the coefficient of any power of $x$ in one side must be equal to that of the same power of $x$ in the other side.*

This comparison of the constant term, leads to the condition:

$$2-D = 3 \rightarrow D = -1.$$  

Now we compare the coefficient of $x^2$:

$$(2+C) = 0 \rightarrow C = -2.$$  

Hence, the required Partial Fraction Form is

$$\frac{x+3}{(x+1)(x-1)} = \frac{2}{x-1} - \frac{1+2x}{1+x^2}.$$  

**Another method to find $C$ and $D$**

Take the red term in the equation ($\sim\sim$), to the LHS, and write:

$$x+3-2(x^2+1) \equiv (Cx+D)(x-1) \rightarrow -(-1+x)(1+2x) = (Cx+D)(x-1)$$

Factorise the LHS.

Then divide by $(x-1)$.

This gives

$$Cx+D \equiv -2x-1.$$  

By comparing coefficients of the like powers, we get $C = -2$ and $D = -1$. We have obtained the same result as we did with the other method.
Example: Obtain the partial fraction form for \( \frac{3x - 1}{(x-2)(x-1)(x^2+1)} \).

We seek the values of the indicated constants, so that the statement below is an identity.

\[
\frac{4x - 3}{(x-2)^2(x^2+1)} \equiv \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{Cx + D}{x^2+1}
\]

From this, clearing the denominators, we obtain that

\[
4x - 3 \equiv A(x-2)(x^2+1) + B(x^2+1) + (Cx + D)(x-2)^2.
\]

Then,

\[
x = 2 \rightarrow 5 = 5B \rightarrow B = 1.
\]

Substituting the value of \( B \) in to the identity above, we get

\[
4x - 3 \equiv A(x-2)(x^2+1) + (x^2+1) + (Cx + D)(x-2)^2.
\]

Comparing the coefficients of \( x^3 \), in both sides, we get

\[
0 = A + C \quad (1)
\]

Now, we compare the constant term, to get,

\[
-3 = -2A + 1 + 4D \rightarrow 2D - A = -2 \quad (2)
\]

By comparing the coefficient of \( x \), we get

\[
4 = A - 4D + 4C \rightarrow A - 4D + 4C = 4 \quad (3)
\]

By solving these three equations simultaneously, we get

\[
C = 0, D = -1, \text{ and } A = 0.
\]

Therefore,

\[
\frac{4x - 3}{(x-2)^2(x^2+1)} \equiv \frac{0}{(x-2)} + \frac{1}{(x-2)^2} + \frac{0 \times x - 1}{x^2 + 1} \equiv \frac{1}{(x-2)^2} - \frac{1}{x^2 + 1}
\]

\[
\equiv \frac{1}{8}.
\]
A Method of Picking up Coefficients Without Expanding

Now, here we copy the identity in (*1), with some annotations.

\[ 4x-3 \equiv A \left( x-2 \right) \left( x^2 + 1 \right) + \left( x^2 + 1 \right) + \frac{Cx}{\text{No } x^3 \text{ terms}} + D \left( x-2 \right)^2. \]  

(*1)

How to pick up the coefficients easily, just by observation? Refer to both the pictorial presentation above, and the explanation below.

In the LHS, there is no \( x^3 \). This means that the coefficient of \( x^3 \) in the LHS is zero. In the RHS, there is \( (A+C)x^3 \). By comparing these coefficients, we get the equation,

\[ A + C = 0. \]

When you do exams, every second is important. Therefore, you need to practice short and visual methods like this one. When you do your homework, keep this in your mind.

Another Method to find A, C and D

Take the red Term in (*1) to its LHS, and obtain:

\[ -\left( x^2 - 4x + 4 \right) \equiv A\left( x-2 \right)\left( x^2 + 1 \right) + (Cx + D)(x - 2)^2 \]

Now, divide the statement above by \( (x - 2) \), and then transfer \( (Cx + D)(x - 2) \) to the LHS, to get (to do this division you need to assume \( x \neq ? \)-Think!):

\[ -(x - 2) - (Cx + D)(x - 2) \equiv A\left( x^2 + 1 \right). \]

By factoring out the common factor \(-(x-2)\) from the Left, we get:

\[ -(x - 2)(Cx + D + 1) \equiv A\left( x^2 + 1 \right) \quad (#2). \]
Comparing the coefficients of \( x^2 \) in (#2), we get
\[-C = A \implies A + C = 0 \quad (a1)\]

Comparison of the constant term in (#2), we get \( 2D + 2 = A \), which leads to
\[2D - A = -2 \quad (a2)\]

Then, we compare the coefficient of \( x \), in (#2), we get \(-D + 2C - 1 = 0\). This gives the condition that
\[D - 2C = -1 \quad (a3)\]

By solving the three equations above, we will get the same partial fraction form, which we calculated with the other method.

**Short Methods with justifications and or Memory Aids**

**Two Linear Factors: CASE I**

Consider the rational expression in the LHS, and its partial fraction decomposition form in the RHS:
\[
\frac{b}{(px+q)(ux+v)} = \frac{A}{px+q} + \frac{B}{ux+v}.
\]

We may calculate the coefficients, \( A \) and \( B \), in the following manner.

- Notice that when we combine the RHS of the statement above, we need the \( x \)-term to be cancelled out and the constant to be equal to \( b \).
- So, first we make the coefficient of \( x \) vanish.
- In order to do so, we put \( p \) (the coefficient of \( x \) which has to vanish) over the first denominator, \( px+q \), and \(-u\) (the coefficient of \( x \) which has to vanish) over the second denominator, \( ux+v \).

*Where we put the negative sign has no magic in it, because if we put \(-p\) over the first denominator, then we need to put \(+u\) over the second denominator, to force the cancellation of the \( x \)-term.*
This will ensure the cancellation of \( x \) in the sum of the two fractions in the RHS.

However, to guarantee that the total numerator in the RHS is equal to \( b \), we use a multiplying factor, \( \frac{b}{R} \), to the whole RHS.

Then we determine the value of \( R \).

\[
\frac{b}{(px+q)(ux+v)} \equiv \frac{b}{R} \left( \frac{p}{px + q} - \frac{u}{ux + v} \right) = \frac{b}{R} \left( \frac{p(px+q)-u(px+q)}{(px + q)(ux + v)} \right)
\]

\[
= \frac{b}{R} \left( \frac{px - px + pv - qu}{(px + q)(ux + v)} \right) = \frac{b}{R} \left( \frac{pv - qu}{(px + q)(ux + v)} \right)
\]

Hence, if we set \( R = pv - qu \), we get the required partial fraction form. That is,

\[
\frac{b}{(px+q)(ux+v)} = \frac{A}{pv-qu} \left( \frac{bp}{px + q} \right) + \frac{B}{pv-qu} \left( \frac{-bu}{ux + v} \right) = \frac{b}{pv-qu} \left( \frac{p}{px + q} - \frac{u}{ux + v} \right).
\]

With

\[
A = \frac{bp}{pv-qu}, \text{ and } B = \frac{-bu}{pv-qu}.
\]

To visually see that how the \( x \) term is cancelled out, follow the red arrows in the diagram below. The South-East Arrow which runs from left to right produces \( pu \). To this, add the term obtained by following the South-West arrow which runs from left to right, \(-pu\).

\[
\frac{b}{(px+q)(ux+v)} \equiv \frac{b}{R} \left( \frac{p}{px + q} + \frac{u}{x + v} \right) = \frac{b}{R} \left( \frac{bR}{(px+q)(ux+v)} \right)
\]

To find the term, \( R \), do the same with the green arrows, Left to Right – Right to Left.
\[
\frac{b}{(px+q)(ux+v)} = \frac{b}{R} \left[ \frac{p}{(p \ x + q)} - \frac{u}{(u \ x + v)} \right]
\]

Here,

\[R = pv - qu.\]

**Two Linear Factors: CASE II**

Now, consider the rational expression and its partial fraction decomposition,

\[
\frac{ax}{(px+q)(ux+v)} = \frac{A}{px+q} + \frac{B}{ux+v}.
\]

This time, when we combine the right hand side of the statement above, we need the \(x\)-term to have the coefficient \(a\), and the constant term to be cancelled out. This can be achieved by putting \(q\) over the denominator, \(px+q\), and \(-v\) over the denominator, \(ux+v\).

\[
\frac{ax}{(px+q)(ux+v)} = \frac{a}{qu-pv} \left[ \frac{q}{(p \ x + q)} - \frac{v}{(u \ x + v)} \right]
\]

This results in the total numerator \(ax(qu-vp)\). Therefore, to make the total numerator equals \(ax\), we divide the result by \((qu-vp)\). The following, diagram helps to visualise this

\[
\frac{ax}{(px+q)(ux+v)} = \frac{a}{R} \left[ \frac{q}{(p \ x + q)} - \frac{v}{(u \ x + v)} \right]
\]

Here,

\[R = qu - vp.\]

**Examples**

**Ex 1**
Apply the arrow method here.

By practice, one can come directly to this final step.

Keeping up with the bad habit of many textbooks, I have used ‘=’ in place of rightful symbol, \( \equiv \). Then come to think of it again, we do not loose much, even if we use just the equal sign, instead of the identity symbol, after we have determined the numerical values of the coefficients.

After some practice, you should be able to apply the arrow method to the original expression, \( \frac{2}{(x-1)(x+3)} \), itself.

**Ex 2**

\[
\frac{5x}{(x-1)(x+3)} = \frac{5x}{(x-1)(x+3)} = \frac{5}{1 \times 3 - 1} \left[ \frac{1}{x-1} - \frac{1}{x+3} \right]
\]

\[
= \frac{2}{4} \left[ \frac{1}{x-1} - \frac{1}{x+3} \right]
\]

\[
= \frac{1}{2(x-1)} - \frac{1}{2(x+3)}.
\]

**Ex 3**

\[
\frac{7}{(8x-5)(2x+3)} = \frac{7}{8 \times 3 - 2 \times 5} \left[ \frac{8}{8x-5} - \frac{2}{2x+3} \right]
\]

\[
= \frac{7}{34} \left[ \frac{8}{8x-5} - \frac{2}{2x+3} \right]
\]

\[
= \frac{28}{17(8x-5)} - \frac{7}{17(2x+3)}.
\]
Ex 4

\[
\frac{9x}{(8x-5)(2x+3)} = \frac{9}{-5 \times 2 - 3 \times 8} \left[ \frac{-5}{8x-5} - \frac{3}{2x+3} \right] = \frac{9}{-34 \times 8x-5 - 3 \times 2x+3} = \frac{45}{34(8x-5)} + \frac{27}{34(2x+3)}.
\]

Two Linear Factors (both x term and constant terms are present in the numerators):
Case III (Combination of Case I and II)

Ex

Find the Partial Fraction Form of \( \frac{2x+3}{(7x-5)(8x+9)} \).

First, we separate the given rational expression, in the following manner.

\[
\frac{2x+3}{(7x-5)(8x+9)} \equiv \frac{2x}{(7x-5)(8x+9)} + \frac{3}{(7x-5)(8x+9)}
\]

Then, we write the first expression in the right in the following manner, where the value of the constant, \( P \), has yet to be determined. Also, note that we have put 5 as the numerator of the first rational expression in the right, and the 9 as the numerator of the second rational in the right, so that when the two expressions are added, the constant term of the numerator vanishes. This cancellation can be visually calculated as \( 5 \times 9 - 5 \times 9 = 0 \).

\[
\frac{2x}{(7x-5)(8x+9)} = P \left( \frac{5}{7x-5} + \frac{9}{8x+9} \right)
\]

When we simplify these two fractions, by adding them, it will produce an \( x \)-term of

\[
P(9 \times 7 + 5 \times 8)x = 103Px.
\]

Since we need this to be equal to \( 2x \), we choose the value of \( P \): \( P = \frac{2}{103} \). Therefore, we have

\[
\frac{2x}{(7x-5)(8x+9)} = \frac{2}{103} \left( \frac{5}{7x-5} + \frac{9}{8x+9} \right)
\]

(F)

Next, we write the rational expression,
in the form,
\[
\frac{3}{(7x-5)(8x+9)} = Q\left(\frac{7}{7x-5} + \frac{-8}{8x+9}\right),
\]

where the value of the constant, \( Q \), has yet to be determined. Also note that we have put 7 as the numerator of the first rational expression in the right and the -8 as the numerator of the second rational in the right, so that when the two expressions are added, the \( x \)-term of the numerator vanishes. This cancellation can be visually calculated as \( 7 \times 8 - 8 \times 7 = 0 \).

When we simplify these two fractions by adding them, it will produce a constant term of
\[
Q(9 \times 7 + 5 \times 8)x = 103Q.
\]

However, we need this to be equal to 3. Thus, we choose the value of \( Q \): \( Q = \frac{3}{103} \).

Therefore, we have
\[
\frac{3}{(7x-5)(8x+9)} = \frac{3}{103}\left(\frac{7}{7x-5} - \frac{8}{8x+9}\right) \quad \text{(G)}
\]

Now, we add (F) and (G), to get
\[
\frac{2x}{(7x-5)(8x+9)} + \frac{3}{(7x-5)(8x+9)} = \frac{2}{103}\left(\frac{5}{7x-5} + \frac{9}{8x+9}\right) + \frac{3}{103}\left(\frac{7}{7x-5} - \frac{8}{8x+9}\right)
\]
\[
= \frac{1}{103}\left(\frac{2 \times 5 + 3 \times 7}{7x-5} + \frac{2 \times 9 - 3 \times 8}{8x+9}\right)
= \frac{1}{103}\left(\frac{31}{7x-5} + \frac{-6}{8x+9}\right)
= \frac{31}{103}\left(\frac{1}{7x-5}\right) - \frac{6}{103}\left(\frac{1}{8x+9}\right)
\]

Two Linear Factors (both x term and constant terms are present in the numerators):
Case III (Determinant Method)

Next, we consider the rational expression and its partial fraction decomposition,
\[
\frac{ax+b}{(px+q)(ux+v)} \equiv \frac{A}{px+q} + \frac{B}{ux+v}.
\]

By writing
\[
\frac{ax+b}{(px+q)(ux+v)} = \frac{ax}{(px+q)(ux+v)} + \frac{b}{(px+q)(ux+v)}
\]
\[
= \frac{A_1}{px+q} + \frac{A_2}{ux+v} + \frac{B_1}{px+q} + \frac{B_2}{ux+v}.
\]

and then applying Case I and II, we can find the constants, \(A_1, A_2, B_1,\) and \(B_2.\) Then, we combine the terms to get the final answer, as demonstrated below.

\[
\frac{ax+b}{(px+q)(ux+v)} \equiv \frac{A_1}{px+q} + \frac{A_2}{ux+v} + \frac{B_1}{px+q} + \frac{B_2}{ux+v}
\]
\[
\equiv \frac{A + B_1}{px+q} + \frac{A_2 + B_2}{ux+v}.
\]

Now, we present a shorter method which can be proved be the approach we just described or solving appropriate simultaneous equations. First we recall the definition of the determinant of a 2×2 matrix.

Let

\[
M = \begin{vmatrix} j & k \\ l & m \end{vmatrix},
\]

be a matrix of dimension of 2×2, where \(j, k, l,\) and \(m\) are real numbers. Then we define

\[
\det \begin{vmatrix} j & k \\ l & m \end{vmatrix} = jm - lk.
\]

Then

\[
\frac{ax+b}{(px+q)(ux+v)} \equiv \frac{A}{px+q} + \frac{B}{ux+v}
\]

With

\[
A = \frac{\det \begin{vmatrix} a & b \\ p & q \end{vmatrix}}{\det \begin{vmatrix} u & v \\ p & q \end{vmatrix}} \quad \text{and} \quad B = \frac{\det \begin{vmatrix} a & b \\ u & v \end{vmatrix}}{\det \begin{vmatrix} p & q \\ u & v \end{vmatrix}}.
\]

That is,
\[
\frac{ax+b}{(px+q)(ux+v)} \equiv \frac{\text{det}\begin{pmatrix} a & b \\ p & q \end{pmatrix}}{px+q} + \frac{\text{det}\begin{pmatrix} u & v \\ p & q \end{pmatrix}}{ux+v}.
\]

How do we remember this? To explore this, let us write

\[\alpha_1 = \text{det}\begin{pmatrix} a & b \\ p & q \end{pmatrix}, \alpha_2 = \text{det}\begin{pmatrix} u & v \\ p & q \end{pmatrix}, \beta_1 = \text{det}\begin{pmatrix} a & b \\ u & v \end{pmatrix}, \text{ and } \beta_2 = \text{det}\begin{pmatrix} p & q \\ u & v \end{pmatrix}.\]

Then,

\[A = \frac{\alpha_1}{\alpha_2} \text{ and } B = \frac{\beta_1}{\beta_2}.\]

This matrix is constructed by replacing the top row of the \(\alpha_1\) matrix with the coefficients of the second factor in the denominator.

Top row: Coefficients of the numerator

Bottom row: Coefficients of the first factor of the denominator
Examples:

Ex 5

\[
\frac{4x + 5}{(3x - 7)(8x - 5)} = \frac{A}{3x - 7} + \frac{B}{8x - 5}
\]

Then

\[
\alpha_1 = \det \begin{pmatrix} 4 & 5 \\ 3 & -7 \end{pmatrix} = 4 \times -7 - 5 \times 3 = -43
\]

\[
\alpha_2 = \det \begin{pmatrix} 8 & -5 \\ 3 & -7 \end{pmatrix} = 8 \times -7 - 5 \times 3 = -41
\]

\[
\beta_1 = \det \begin{pmatrix} 4 & 5 \\ 8 & -5 \end{pmatrix} = 4 \times -5 - 5 \times 8 = -60
\]

\[
\beta_2 = \det \begin{pmatrix} 3 & -7 \\ 8 & -5 \end{pmatrix} = 3 \times -5 - 7 \times 8 = 41
\]

\[
\Rightarrow A = \frac{\alpha_1}{\alpha_2} = \frac{43}{41}, \quad \text{and} \quad B = \frac{\beta_1}{\beta_2} = \frac{-60}{41},
\]

Or

\[
\frac{4x + 5}{(3x - 7)(8x - 5)} = \frac{4 \times -7 - 5 \times 3}{(3x - 7)} + \frac{4 \times -5 - 5 \times 8}{(8x - 5)}
\]

\[
= \frac{43}{41(3x - 7)} - \frac{60}{41(8x - 5)}
\]
Ex 6

\[
\frac{3x + 7}{(4x - 5)(9x - 2)} = \frac{A}{4x - 5} + \frac{B}{9x - 2}
\]

Then

\[
\alpha_1 = \det \begin{pmatrix} 3 & 7 \\ 4 & -5 \end{pmatrix} = 3 \times -5 - 7 \times 4 = -43
\]

\[
\alpha_2 = \det \begin{pmatrix} 9 & -2 \\ 4 & -5 \end{pmatrix} = 9 \times -5 - 2 \times 4 = -37
\]

\[
\beta_1 = \det \begin{pmatrix} 3 & 7 \\ 9 & -2 \end{pmatrix} = 3 \times -2 - 7 \times 9 = -69
\]

\[
\beta_2 = \det \begin{pmatrix} 4 & -5 \\ 9 & -2 \end{pmatrix} = 4 \times -2 - 5 \times 9 = 37
\]

Or

\[
\frac{3x + 7}{(4x - 5)(9x - 2)} = \frac{3 \times -5 - 7 \times 4}{9 \times -5 - (-2) \times 4} \cdot \frac{3 \times -2 - 7 \times 9}{-4 \times -2 - (-5) \times 9}
\]

\[
= \frac{-43}{-37(3x - 7)} + \frac{-69}{37(8x - 5)}
\]

\[
= \frac{43}{37(3x - 7)} - \frac{69}{37(8x - 5)}.
\]

CASE IV (Repeated Factor)

Consider the Partial Fraction Form of \( \frac{ax + b}{(px + q)^2} \). This has to be formulated as:

\[
\frac{ax + b}{(px + q)^2} \equiv \frac{A}{px + q} + \frac{B}{(px + q)^2}.
\]
\begin{align*}
\frac{ax+b}{(px+q)^2} &= \frac{A}{px+q} + \frac{B}{(px+q)^2} \\
&= \frac{A(px+q) + B}{(px+q)^2} = \frac{Ap x + Aq + B}{(px+q)^2} \\
\frac{ax+b}{(px+q)^2} &= \frac{a}{p(px+q)} + \frac{bp-aq}{p(px+q)^2}.
\end{align*}

Notice that when we process the RHS, the $x$-term comes only from the Term, $A(px+q)$.

This gives that $Ap x = ax$. Thus,

\[ Ap = a \implies A = \frac{a}{p} \quad \text{and} \quad Aq = \frac{aq}{p}. \]

This adds an extra constant term, $\frac{aq}{p}$.

Therefore, since $b = Aq + B$

\[ B = b - Aq = b - \frac{aq}{p} = \frac{bp-aq}{p}. \]

**Examples**

When you do the following examples, use the reasoning contained in the red-outlined textbox.

**Ex 7**

\[ \frac{4x+3}{(5x-7)^2} = \frac{4}{5(5x-7)} + \frac{3 \times 5 - (4 \times 7)}{5(5x-7)^2} = \frac{4}{5(5x-7)} + \frac{43}{5(5x-7)^2} \]

**Ex 8**

\[ \frac{5x-2}{(7x-9)^2} = \frac{5}{7(7x-9)} + \frac{-2 \times 7 - (5 \times 9)}{7(7x-9)^2} = \frac{5}{7(7x-9)} + \frac{31}{7(7x-9)^2} \]
**How to remember this?**

To obtain the value of $A$, follow the reasoning contained in the red-outlined textbox. To calculate $B$, use

$$\frac{ax+b}{(px+q)^2} \Rightarrow \frac{\text{blue-top} \times \text{blue bottom}}{\text{blue-bottom}} - \frac{\text{brown-top} \times \text{brown-bottom}}{\text{blue-bottom}} = \frac{bp - aq}{p}.$$

For other forms of partial fractions, it appears that the standard procedures are more suitable, which has been previously discussed in detail.

**Remark:** To remember these techniques, remembering steps is not convenient. However, if you remember the reasoning behind these steps of each of the cases, then you can remember these very quick methods much more easily.